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Some structures and notation of Q-analysis

J H Johnson¶
Department of Geography, University of Cambridge, Cambridge CB2 3EN, England
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Abstract. This paper is intended to be a short and convenient reference for some of the standard definitions and notation of Q-analysis. Notation is established for relations between sets; simplicial complexes determined by relations; q-connectivity and Q-analysis; the structure vectors and methods of comparing them; antivertices; patterns on a complex; chain complexes and pattern polynomials; the face and coface operators; t-forces; backcloth and traffic on the backcloth; slicing; and strain pairs. The paper contains discussions of structure vector comparisons, the relationship between the structure vectors and the defining relation, and strain pairs. The notation and definitions are consistent with Atkin's standard texts, and the paper is self-contained.

Introduction

This paper is intended to provide a short and convenient reference for some of the standard definitions and notation of Q-analysis. Q-analysis has been developed by Atkin and co-workers over a decade or more, and the standard references are Atkin's books (Atkin, 1974b; 1977; 1981), and papers (Atkin, 1974c; 1974d; 1975). This paper attempts to summarise some of the structures which have been useful in applications and to develop a consistent notation. All the definitions and notation are consistent with Atkin's work, but a number of notational extensions are given. To make this presentation as useful as possible any deviations from Atkin's notation are explicitly noted.

Relations

Consider two finite sets A and B. Let $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_n\}$. Let μ be a relation between A and B, that is, for every $a_i \in A$ and $b_j \in B$ there is a rule to decide if a_i is μ-related to b_j. A very clear presentation of relations in terms of propositional functions may be found in Lipschutz (1964). The notation $a_i \mu b_j$ means a_i is μ-related to b_j. The *complementary relation* not-μ is written symbolically $\tilde{\mu}$ (or sometimes $\sim\mu$) and defined as: $a_i \tilde{\mu} b_j$ if and only if a_i is not μ-related to b_j. Usually relations have the property that either $a_i \mu b_j$ or $a_i \tilde{\mu} b_j$ (but not both) for every $a_i \in A$ and $b_j \in B$, this being the 'law of the excluded middle'.

Every relation determines a unique subset of $A \times B$ called its *solution set.* This is denoted by μ^* and is defined by the rule

$$(a_i, b_j) \in \mu^* \subseteq A \times B , \qquad \text{iff } a_i \mu b_j .$$

By an abuse of language it is common to use the symbol μ both for the relation and for its solution set and to write $\mu \subseteq A \times B$. The logical distinction between the relation and its solution set is usually clear from context. The explicit notational distinction between μ and μ^* allows discrimination between relations with different intentional meaning but the same solution set (Bandler and Johnson, 1972). If μ and and ν are relations it is sometimes an advantage to allow $\mu \neq \nu$ when $\mu^* = \nu^*$; the observation of equal solution sets establishes a relationship between the propositions

¶ Present address: Centre for Configurational Studies, Design Discipline, Faculty of Technology, The Open University, Milton Keynes, MK7 6AA, England.

defining the relations, but does not require that the propositions have the same meaning. The rule for defining μ^* means that any subset of $A \times B$ can be interpreted as a solution set and thus defines a relation between A and B. This method of defining relations is extensional and well suited to formal mathematical presentations, and the abuse of language by using μ both for the relation and for its solution set is invariably used. In this presentation μ^* will not be used.

The inverse relation of μ is denoted μ^{-1}. It is defined as $\mu^{-1} \subseteq B \times A$ with $(b_j, a_i) \in \mu^{-1}$ if and only if $(a_i, b_j) \in \mu$. In the special case that μ is a 1–1 mapping, μ^{-1} is the usual inverse mapping.

A relation $\mu \subseteq A \times B$ can be represented by an *incidence matrix* $\mathrm{M} = [M_{ij}]$, where

$$M_{ij} = \begin{cases} 1, & \text{iff } a_i \text{ is } \mu\text{-related to } b_j, \\ 0, & \text{otherwise.} \end{cases}$$

If M is the incidence matrix of μ it follows that its transpose, denoted by M^T, is the incidence matrix of the inverse relation μ^{-1}. For this reason μ^{-1} is sometimes written as μ^T and called the *transpose relation*. Every relation has a unique incidence matrix and every incidence matrix has a unique (extensional) relation.

By a similar abuse of language to that which uses the symbol μ to represent both the relation and its solution set, it is sometimes convenient to use the symbol μ to represent both the relation and its incidence matrix M. The notation is then extended so that μ_i represents the ith row of the incidence matrix and μ_j^T represents the jth column. The appropriate interpretation for μ is either given explicitly or is clear from the context.

A matrix $\check{\mathrm{M}}$ with arbitrary entries can be made into an incidence matrix with entries 1 or 0 by a process of *slicing*. In its most general form slicing is a mapping (usually nonlinear) on the set of $m \times n$ matrices. For all practical purposes it suffices to consider the set of $m \times n$ matrices with integer entries, the symbol J being used to represent the integers. Let s be a mapping, s: $J \to \{0, 1\}$, called the *slicing mapping*. Usually s is defined in terms of so-called *slicing parameters* α and β where

$$\mathrm{s}(\check{M}_{ij}) = \begin{cases} 1, & \text{iff } \alpha \leqslant \check{M}_{ij} \leqslant \beta, \\ 0, & \text{otherwise.} \end{cases}$$

The *slicing mapping* S: $\check{\mathrm{M}} \to \mathrm{M}$ is given by $[M_{ij}] = [\mathrm{s}(\check{M}_{ij})]$. It is possible to define a slicing mapping s_i for each row, a slicing mapping s_j for each column, or even a slicing mapping s_{ij} for each matrix entry. The slicing can then be extended as $[M_{ij}] = [\mathrm{s}_i(\check{M}_{ij})]$, $[M_{ij}] = [\mathrm{s}_j(\check{M}_{ij})]$, or $[M_{ij}] = [\mathrm{s}_{ij}(\check{M}_{ij})]$, respectively. The use and interpretation of slicing depend on the particular application. In Q-analysis the meaning is usually in terms of sliced traffic values on a restriction of the backcloth, these terms being defined later in this paper.

Simplicial complexes determined by a relation

Let $V = \{v_1, ..., v_n\}$ be a set with elements called *vertices*. Let $\{v_{\alpha_0}, v_{\alpha_1}, ..., v_{\alpha_p}\}$ be a subset of V. Any such subset determines an object written

$$\sigma_p = \langle v_{\alpha_0}, v_{\alpha_1}, ..., v_{\alpha_p} \rangle$$

which is called a *p-simplex*. In general more than one p-simplex is under investigation and the notation σ_p^i means the ith p-simplex. A simplex with $p+1$ distinct vertices is said to have *dimension p*.

Every p-simplex can be represented as a polyhedron in p-dimensional space, a 0-simplex being a point, a 1-simplex being a line, a 2-simplex being a triangle, and so on (see figure 1). This is why a p-simplex has $p+1$ vertices.

Simplices have different properties from the sets of vertices which define them and the angular brackets emphasise the simplex is not a set as such. For example, in set theory $\{v_3, v_7, v_3\} = \{v_3, v_7\}$, but a similar identification for simplices is undesirable; in general $\langle v_3, v_7, v_3\rangle \neq \langle v_3, v_7\rangle$.

Much of the terminology and notation of Q-analysis comes from algebraic topology (which does not mean that an understanding of algebraic topology is necessary to study Q-analysis). In particular the subscript-superscript notation follows that of Hilton and Wylie (1965), the subscript p of the symbol σ_p^i being the dimension of the simplex and the superscript i being the index used to discriminate the various p-simplices.

Consider two simplices σ_p and σ_q, $\sigma_p = \{v_{\alpha_0}, v_{\alpha_1}, ..., v_{\alpha_p}\}$ and $\sigma_q = \{v_{\beta_0}, v_{\beta_1}, ..., v_{\beta_q}\}$. The simplex σ_q is said to be a *q-dimensional face* of σ_p, or q-face of σ_p, if and only if every vertex of σ_q is also a vertex of σ_p. The notation $\sigma_q \lesssim \sigma_p$ means σ_q is a face of σ_p. In general

$$\sigma_q \lesssim \sigma_p, \quad \text{iff} \quad \{v_{\beta_0}, v_{\beta_1}, ..., v_{\beta_q}\} \subseteq \{v_{\alpha_0}, v_{\alpha_1}, ..., v_{\alpha_p}\}.$$

The *face relation* $\lesssim$ so defined is a partial order on any set of simplices with vertex set V.

Let K be a set of simplices of mixed dimensions. K is a *simplicial complex* if and only if $\sigma_p \in K$ and $\sigma_q \lesssim \sigma_p$ imply $\sigma_q \in K$. Two simplicial complexes can be constructed from a relation μ, $\mu \subseteq A \times B$ as follows. Let B be the vertex set for the first complex. The set of simplices $K_A(B, \mu)$ is defined as follows: $\langle b_{\alpha_0}, b_{\alpha_1}, ..., b_{\alpha_p}\rangle$ belongs to $K_A(B, \mu)$ iff there exists $a_i \in A$ such that a_i is μ-related to b_j for $j = \alpha_0, \alpha_1, ..., \alpha_p$.

The notation $\sigma(a_i)$ represents the simplex with vertices all those b_j which are μ-related to a_i. If it is known that a_i is μ-related to exactly $p+1$ distinct b_j the notation is extended to $\sigma_p(a_i)$. In other words $\sigma_p(a_i) = \langle b_{\alpha_0}, b_{\alpha_1}, ..., b_{\alpha_p}\rangle$, where $a_i \mu b_j$ iff $j \in \{\alpha_0, \alpha_1, ..., \alpha_p\}$; and a_i is called the *name* (possibly among many) of the simplex $\sigma(a_i)$. This notation is consistent with the definition of the *sigma-mapping* σ: $A \to K_A(B, \mu)$ defined as σ: $a_i \to \sigma(a_i)$. This makes it clear that the elements of A and the simplices of $K_A(B, \mu)$ are different entities. In general the sigma-mapping is many to one and into. The sigma-mapping notation does not appear in Atkin's standard texts, but sometimes it gives clarity in complicated applications.

Let σ_q be a simplex of $K_A(B, \mu)$ with $\sigma_q = \langle b_{\alpha_0}, b_{\alpha_1}, ..., b_{\alpha_q}\rangle$. By definition this means there exists $a_i \in A$ which is μ-related to each of the b_{α_j}. Consider a simplex σ_r which is a face of σ_q, $\sigma_r \lesssim \sigma_q$. Let $\sigma_r = \langle b_{\beta_0}, b_{\beta_1}, ..., b_{\beta_r}\rangle$. By definition of the face

Number of vertices	Dimension	Symbol and common name	Geometric representation
1	0	σ_0 point	
2	1	σ_1 line	
3	2	σ_2 triangle	
4	3	σ_3 tetrahedron	
5	4	σ_4	
6	5	σ_5	

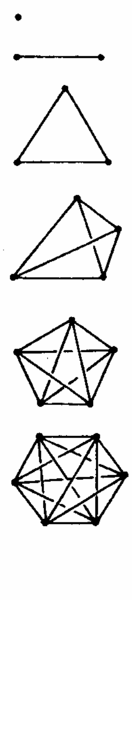

Figure 1. Simplices as polyhedra.

relation $\{b_{\beta_0}, b_{\beta_1}, ..., b_{\beta_r}\} \subseteq \{b_{\alpha_0}, b_{\alpha_1}, ..., b_{\alpha_q}\}$. Thus a_i is also related to each of the b_{β_j}, and hence $\sigma_r \in K_A(B, \mu)$. Thus if σ_q is a simplex of $K_A(B, \mu)$, every face of σ_q is also a simplex of $K_A(B, \mu)$, and $K_A(B, \mu)$ is a simplicial complex.

The vertices of $K_A(B, \mu)$ can be identified with the 0-dimensional simplices:

$$\sigma_0^j = \langle b_j \rangle , \qquad \text{for } j = 1, ..., n .$$

The largest value of p for which there exists $\sigma_p \in K_A(B, \mu)$ is said to be the *dimension* of $K_A(B, \mu)$.

This completes the construction of the first simplicial complex from the relation μ, $\mu \subseteq A \times B$. The second simplicial complex is constructed in just the same manner for the inverse relation $\mu^{-1} \subseteq B \times A$. This complex is written $K_B(A, \mu^{-1})$ and is called the *conjugate complex* of $K_A(B, \mu)$.

When there can be no ambiguity $K_A(B, \mu)$ is abbreviated to K_A and $K_B(A, \mu^{-1})$ is abbreviated to K_B.

The vertices of the simplex $\sigma(a_i)$ in K_A may be identified by looking along the ith row of the incidence matrix representing μ, so that b_j is a vertex of $\sigma(a_i)$ iff there is a 1 in the jth column. Similarly for K_B, the simplex $\sigma(b_j)$ has the vertex a_i iff there is a 1 in the ith row of the jth column of the incidence matrix.

An example

Let figure 2 represent a T-junction between two roads. The intersection has been divided into four 'links' labelled L_1, L_2, L_3, and L_4. There are three points of entry into the junction, a, c, and e, and three points of departure, b, d, and f. There are six 'routes' through the junction traversing various sets of links summarised in the incidence matrix table 1. The relation λ, where $\lambda \subseteq R \times L$ is defined as: R_i is λ-related to L_j iff R_i traverses L_j.

The complex $K_R(L, \lambda)$ contains the named simplices

$$\sigma_1(\text{ab}) = \langle L_1, L_2 \rangle , \quad \sigma_2(\text{ad}) = \langle L_1, L_2, L_3 \rangle , \quad \sigma_0(\text{cd}) = \langle L_3 \rangle ,$$

$$\sigma_1(\text{cf}) = \langle L_3, L_4 \rangle , \quad \sigma_0(\text{ef}) = \langle L_4 \rangle , \quad \sigma_2(\text{eb}) = \langle L_1, L_2, L_4 \rangle ,$$

and their unnamed faces

$$\langle L_1, L_3 \rangle , \quad \langle L_1, L_4 \rangle , \quad \langle L_2, L_3 \rangle , \quad \langle L_2, L_4 \rangle , \quad \langle L_1 \rangle , \quad \langle L_2 \rangle .$$

In this example $\sigma_1(\text{ab}) \lesssim \sigma_2(\text{ad})$, $\sigma_1(\text{ab}) \lesssim \sigma_2(\text{eb})$, $\sigma_0(\text{cd}) \lesssim \sigma_2(\text{ad})$, $\sigma_0(\text{cd}) \lesssim \sigma_1(\text{cf})$ $\sigma_0(\text{ef}) \lesssim \sigma_1(\text{cf})$, and $\sigma_0(\text{ef}) \lesssim \sigma_2(\text{eb})$.

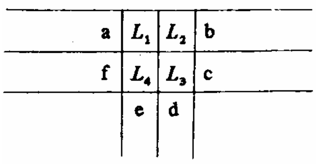

Figure 2. The T-junction of two roads.

Table 1. The incidence matrix of λ, $\lambda \subseteq R \times L$.

λ	L_1	L_2	L_3	L_4
ab	1	1	0	0
ad	1	1	1	0
cd	0	0	1	0
cf	0	0	1	1
ef	0	0	0	1
eb	1	1	0	1

The complex $K_L(R, \lambda^{-1})$ contains the named simplices

$\sigma_2(L_1) = \langle \mathrm{ab, ad, eb} \rangle$, $\quad \sigma_2(L_2) = \langle \mathrm{ab, ad, eb} \rangle$,

$\sigma_2(L_3) = \langle \mathrm{ad, cd, cf} \rangle$, $\quad \sigma_2(L_4) = \langle \mathrm{cf, ef, eb} \rangle$,

and their unnamed faces

$\langle \mathrm{ab, ad} \rangle$, $\langle \mathrm{ab, eb} \rangle$, $\langle \mathrm{ad, eb} \rangle$, $\langle \mathrm{ad, cd} \rangle$, $\langle \mathrm{ad, cf} \rangle$, $\langle \mathrm{cd, cf} \rangle$,

$\langle \mathrm{cf, ef} \rangle$, $\langle \mathrm{cf, eb} \rangle$, $\langle \mathrm{ef, eb} \rangle$, $\langle \mathrm{ab} \rangle$, $\langle \mathrm{ad} \rangle$, $\langle \mathrm{cd} \rangle$,

$\langle \mathrm{cf} \rangle$, $\langle \mathrm{ef} \rangle$, $\langle \mathrm{eb} \rangle$.

In this example $\sigma_2(L_1) = \sigma_2(L_2)$, but clearly $L_1 \neq L_2$.

Every simplicial complex has a *geometric representation* as a collection of convex polyhedra in a euclidean space E^H. It has been demonstrated that the value of H for this representation is at most $2N+1$ when the complex has dimension N (Hilton and Wylie (1965). The geometric representations for $K_R(L, \lambda)$ and $K_L(R, \lambda^{-1})$ are given in figure 3.

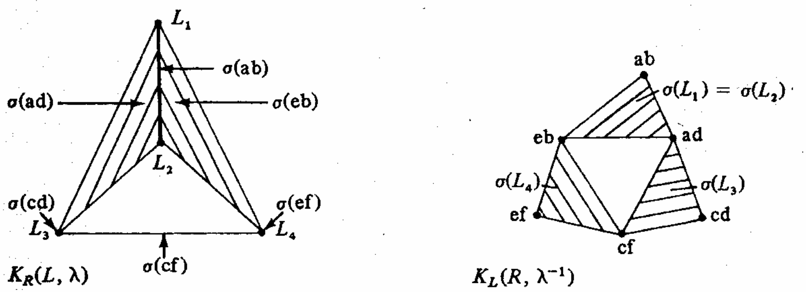

Figure 3. Geometric representations of simplicial complexes.

Chains of q-connection

Two simplices are said to be *q-near* in K iff they share a q-dimensional face in K. Two simplices σ_p and σ_r in a complex K are joined by a *chain of q-connection* iff there exists a finite sequence of simplices $\sigma_{\alpha_1}, ..., \sigma_{\alpha_h}$, such that (1) $\sigma_{\alpha_1} \lesssim \sigma_p$, (2) $\sigma_{\alpha_h} \lesssim \sigma_r$, and (3) σ_{α_i} and $\sigma_{\alpha_{i+1}}$ share a common face as σ_{β_i}, $i = 1, ..., h-1$, where the least of the integers $\alpha_1, \beta_1, ..., \beta_{h-1}, \alpha_h$ has value q or more.

The sequence of simplices $\sigma_{\alpha_1}, ..., \sigma_{\alpha_h}$ is called a *chain of q-connection*, or *q-connectivity*, and it is said to have length $(h-1)$. When a q-connectivity exists between σ_p and σ_r, that is, there exists an intermediate sequence of q-near simplices, σ_p and σ_r are said to be *q-connected*. Every p-simplex is p-connected to itself by a chain of length zero. If σ_p and σ_r are q-connected they are also $(q-1)$-connected for $(q-1) \geqslant 0$.

The number $\check{q}$ (the *bottom-q*) associated with a simplex σ_r is the greatest value of q for which σ_r is q-connected to a distinct simplex in K. The number $\hat{q}$ (the *top-q*) associated with each simplex σ_r is the dimension of the simplex, in this case $\hat{q} = r$. The *eccentricity* of the simplex σ_r is defined to be the rational number given by $\mathrm{ecc}(\sigma_r) = (\hat{q}-\check{q})/(\check{q}+1)$. This definition of eccentricity accords with intuition: $\check{q}+1$ is the greatest number of vertices σ_r shares with any simplex, and $\hat{q}-\check{q}$ is the number of vertices making σ_r different from that simplex. Thus $(\hat{q}-\check{q})/(\check{q}+1)$ is a measure of the individuality, otherwise eccentricity, of σ_r.

The relation 'is q-connected to' on the simplices of a complex K is reflexive, symmetric, and transitive, that is, it is an equivalence relation which will be denoted by γ_q. Let K_q be the set of simplices in K with dimension greater than or equal to q. Then γ_q partitions K_q into equivalence classes of q-connected simplices. These equivalence classes are members of the quotient set K_q/γ_q and they are called the *q-connected components of K.* Every simplex in a q-component is q-connected to every other simplex in that component, but no simplex in one q-component is q-connected to any simplex in a distinct q-connected component.

The cardinality of K_q/γ_q is denoted by Q_q and is the number of distinct q-connected components in K. The determination of the components for each value of q, and the listing of the numbers $Q_0, Q_1, ..., Q_N$, where $N = \dim K$, is termed a *Q-analysis* of K. The vector $(Q_0, Q_1, ..., Q_N)$ is called the *first structure vector of K* and is denoted by Q.

The computer programmes for Q-analysis usually list the named simplices in the q-connected components on the assumption that all unnamed faces are implicitly in the components at the appropriate q-values.

An example

The Q-analysis of the T-junction structures is given as:

$K_R(L, \lambda)$	$K_L(R, \lambda^{-1})$
$q = 2$	$q = 2$
$(\sigma_2(\text{ad})), (\sigma_2(\text{eb}))$	$(\sigma_2(L_1), \sigma_2(L_2)), (\sigma_2(L_3)), (\sigma_2(L_4))$
$q = 1$	$q = 1$
$(\sigma_2(\text{ad}), \sigma_2(\text{eb}), \sigma_1(\text{ab})), (\sigma_1(\text{cf}))$	$(\sigma_2(L_1), \sigma_2(L_2)), (\sigma_2(L_3)), (\sigma_2(L_4))$
$q = 0$	$q = 0$
$(\sigma_2(\text{ad}), \sigma_2(\text{eb}), \sigma_1(\text{ab}), \sigma_1(\text{cf}), \sigma_0(\text{cd}), \sigma_0(\text{ef}))$	$(\sigma_2(L_1), \sigma_2(L_2), \sigma_2(L_3), \sigma_2(L_4))$
$Q = (\overset{0}{1}, \overset{1}{2}, \overset{2}{2})$	$Q = (\overset{0}{1}, \overset{1}{3}, \overset{2}{3})$

As noted previously $\sigma_2(L_1) = \sigma_2(L_2)$, so the first of the two-dimensional components in K_L contains only a single distinct simplex. This suggests a new link be defined for the union of L_1 and L_2 as pieces of junctions. In this case the number of vertices of $K_R(L, \lambda)$ is reduced by one, and this will change the structure of $K_R(L, \lambda)$ accordingly. The merits of redefining sets depends on the particular application and this particular case will not be further discussed here.

The first structure vector Q provides a measure of some of the global structure of K in that it gives the number of q-connected components for each K, but it overlooks the internal structure of the components of K. Q_0 corresponds to the zeroth Betti number of the complex and it has been shown that $K_A(B, \mu)$ and $K_B(A, \mu^{-1})$ have the same 0-Betti number (Dowker, 1952). Q_0 is the number of arcwise connected pieces of the complex K, and when Q_0 is greater than 1 the complex exists as Q_0 disconnected pieces. In this case the structure vector is the sum of the structure vectors of the Q_0 separate subcomplexes of K. Without loss of generality this paper considers only complexes with $Q_0 = 1$.

Let U be a unit vector with N entries all equal to 1:

$$U = (1, ..., 1) .$$

The *obstruction vector* of a complex K is denoted by $\hat{Q}$. If $\dim K = N$, it is defined as

$$\hat{Q} = Q - U, \quad \text{or} \quad \hat{Q} = (Q_0 - 1, Q_1 - 1, ..., Q_N - 1) .$$

The obstruction vector indicates the obstruction to changes in various dimensionally graded mappings defined on the complex. Given any q-connected component, then, intuitively, there are exactly $Q_q - 1$ q-dimensional 'gaps' between it and the other components. The notation here varies slightly from Atkin's: here the vectors are enclosed by round brackets (...) rather than curly brackets {...}, and the vectors are written with the first entry for $q = 0$, the second for $q = 1$, etc, instead of the first entry for $q = N$, the second for $q = N-1$, and so on. Usually the dimensions involved are written above the value Q_q, for example, the obstruction vector for $K_R(L, \lambda)$ would be written as $\hat{Q} = (\overset{0}{0}, \overset{1}{1}, \overset{2}{1})$.

It has been useful to define a *second structure vector* $\boldsymbol{P}$ where the entry P_q is the number of named simplices in the complex K with dimension greater than or equal to q (Johnson, 1976).

Comparing the structure of complexes by their structure vectors

Let Q and Q' be the structure vectors of two complexes given by $K = K_A(B, \mu)$ and $K' = K_A(B, \mu')$. Suppose there is a dimension p such that $Q_q \geqslant Q'_q$ for $q \geqslant p$, and $Q_q \leqslant Q'_q$ for $q < p$. For high values of q greater than p, $Q_q > Q'_q$ will usually be associated with some of the $\sigma(a_i)$ having greater dimension, and hence being q-connected at higher values of q, in K than K'. For low values of q less than p, $Q_q < Q'_q$ will usually mean more pairs $\sigma(a_i)$ and $\sigma(a_j)$ are not q-connected (and hence not $q+1$, not $q+2$ connected, etc) in K' than in K. This motivates the definition of a comparison of the connectivity structures of two complexes based on their structure vectors (Johnson, 1976).

If $\dim K' \leqslant \dim K$ the structure vectors are made of equal length by defining $Q'_q = 0$ and $P'_q = 0$ for $\dim K \geqslant q > \dim K'$. The *flipover value* of Q with respect to Q' is defined when it exists as the lowest value of p such that $Q_q \geqslant Q'_q$ for q greater or equal to p, and $Q_q \leqslant Q'_q$ for q less than p. If $Q \neq Q'$ and there exists a flipover value of Q with respect to Q' the notation $Q' < Q$ and $K' <_Q K$ is used. $K' <_Q K$ is interpreted as a crude indication that more pairs of simplices are q-connected in K than in K' for each q-value.

Let $\boldsymbol{P}$ and $\boldsymbol{P}'$ be the second structure vectors of K and K' respectively. The notation $\boldsymbol{P}' < \boldsymbol{P}$ and $K' <_P K$ will be used when $\boldsymbol{P} \neq \boldsymbol{P}'$ and $P_q \geqslant P'_q$ for all values of q. $K' <_P K$ is interpreted as meaning there are as many or more q-dimensional simplices in K than in K' for each q-value. A weaker dimension comparison to allow for the introduction of low dimensional simplices in K' may be achieved by defining the flipover value, r, of $\boldsymbol{P}$ with respect to $\boldsymbol{P}'$ as follows. The *flipover value*, r, of $\boldsymbol{P}$ with respect to $\boldsymbol{P}'$ when it exists is the lowest value of q such that $P_q \geqslant P'_q$ for q greater or equal to r, and $P_q \leqslant P'_q$ for q less than r. If $\boldsymbol{P} \neq \boldsymbol{P}'$ and there exists a flipover value of $\boldsymbol{P}$ with respect to $\boldsymbol{P}'$ the notation $\boldsymbol{P}' \lesssim \boldsymbol{P}$ and $K' \lesssim_P K$ is used. The flipover value of $\boldsymbol{P}$ with respect to $\boldsymbol{P}'$ may be different from that of Q with respect to Q', and either may exist without the other.

Neither the second structure vector nor the structure vector comparisons are to be found in the standard texts on Q-analysis. However, they have been used to compare the structure of road intersections and to show this affects the road traffic flows they can accommodate (Johnson, 1976).

Antivertices

In applications the observation that a_i is not related to b_j may be as important as the observation that a_i is related to b_j. Such negative observations can be explicitly recorded and integrated into the structure by use of the antivertices. A full description and definition of antivertices requires consideration of the extended exterior algebra (Atkin, 1974a; 1977) but the following will do for many practical purposes.

Let μ be a relation between the sets A and B with incidence matrix $\mathbf{M}$. Let $\tilde{\mu}$ be its complementary relation, $a_i \tilde{\mu} b_j$ iff a_i is not μ-related to b_j, with incidence matrix $\tilde{\mathbf{M}}$. For each a_i define an *antivertex* $\tilde{a}_i$ and let $\tilde{A}$ be the set of all these antivertices. Similarly for each b_j define an antivertex $\tilde{b}_j$ and let $\tilde{B}$ be the set of all these antivertices. The relation μ can be extended to the relation denoted $\mu^+ \subseteq (A \cup \tilde{A}) \times (B \cup \tilde{B})$ by the definitions:

$a_i \mu^+ b_j$ iff $a_i \mu b_j$

$\tilde{a}_i \mu^+ b_j$ iff $a_i \tilde{\mu} b_j$

$a_i \mu^+ \tilde{b}_j$ iff $a_i \tilde{\mu} b_j$

$\tilde{a}_i \mu^+ \tilde{b}_j$ iff $a_i \mu b_j$.

Let $A^+ = (A \cup \tilde{A})$ and $B^+ = (B \cup \tilde{B})$. Then the incidence matrix of $\mu^+ \subseteq A^+ \times B^+$ has the form

$$\mathbf{M}^+ = \begin{bmatrix} \mathbf{M} & \tilde{\mathbf{M}} \\ \tilde{\mathbf{M}} & \mathbf{M} \end{bmatrix}.$$

It is the case that all $\sigma(a_i)$ and $\sigma(\tilde{a}_i)$ have the same dimension, $n-1$, in $K_A^+(B^+, \mu^+)$ when M has n columns. Similarly all the $\sigma(b_j)$ and $\sigma(\tilde{b}_j)$ have the same dimension in $K_B^+(A^+, \mu^+)$, viz $m-1$ where M has m rows. It has been suggested that the antivertices have an important role to play in taxonomy and classification (Johnson, 1978; 1981).

Mathematical problems in Q-analysis

It is easy to find examples of distinct simplicial complexes which have the same structure vector. However, there is an interesting question first posed by Atkin which may be paraphrased as "under what conditions, if any, do the structure vectors of K_A and K_B determine the relation $\mu \subseteq A \times B$". A discussion of this problem (Johnson, 1977) can be summarised as follows.

Let Q_A and Q_B be the first structure vectors of $K_A(B, \mu)$ and $K_B(A, \mu^{-1})$, respectively.

Question 1

Given the structure vectors Q_A and Q_B can the relation $\mu \subseteq A \times B$ be reconstructed?

It has been shown that for every strictly positive integer vector V there exists at least one simplicial complex K with structure vector Q such that $V = Q$. In other words every such vector V is the structure vector of at least one simplicial complex. This suggests:

Question 2

Given the strictly positive integer vectors V and V' is there a (unique) relation $\mu \subseteq A \times B$ such that $V = Q_A$ and $V' = Q_B$?

It is clear this question can only be answered up to some kind of equivalence of relations in which the particular names of the simplices are ignored. Permutations of the rows and columns of the incidence matrix will not alter the structure vectors of K_A and K_B, but simply give their simplices different names and permute their vertices. The kind of isomorphism relevant to this discussion can be defined in terms of face-saving maps (Atkin, 1977). A mapping Ψ: $K_A \rightarrow K'_A$ is a *face-saving map* iff σ_r q-near σ_s in K_A implies $\Psi(\sigma_r)$ is q-near $\Psi(\sigma_s)$ in K'_A. If Ψ is one to one and onto and its inverse is also face-saving it will be called a *face-saving isomorphism*. Thus the relation μ of question 2 can at best be determined up to face-saving isomorphism.

Let α and β be permutation matrices such that $\alpha\mathbf{M}\beta$ is the matrix M with rows and columns permuted. By the previous discussion the complexes obtained from these matrices are face-saving isomorphic and they have the same structure vectors. The essence of question 2 lies in asking if there is a matrix $\mathbf{M}^*$ where for all permutation matrices $\mathbf{M}^* \neq \alpha\mathbf{M}\beta$, but M and $\mathbf{M}^*$ have the same structure vectors. This question is unanswered at the moment.

It may be that Q_A and Q_B alone do not characterise the relation μ by themselves, but the extended complex may suffice. Therefore the question could be rephrased as follows:

Question 3
Is μ^+ and hence μ characterised by Q_A^+ up to face-saving isomorphism, is μ^+ and hence μ characterised by Q_B^+ up to face-saving isomorphism; and is μ^+ and hence μ characterised by Q_A and Q_B up to face-saving isomorphism?

These problems can be seen as more than mathematical curiosities: whatever their answers there exists a class of simplicial complexes (and defining relations) for each structure vector. This means the set of all complexes (relations) can be classified by the structure vectors, a classification based entirely on the concept of connectivity. There are many examples in the literature of the practical significance of multidimensional structure and some special results for particular complexes. The possibility of appealing to a substantial and coherent body of results within a classification when studying data relations is very attractive.

These structural classification problems can be extended by consideration of slicing: given a particular integer matrix there are many incidence matrices which can be obtained by slicing. The relationships between the structures obtained by the various slicings also merit investigation.

Patterns on a complex
Let K be a simplicial complex. Let J be a set called the *coefficient set* which will be assumed to be the integers or rationals in this paper. In general the coefficient set can be anything, but usually it must have algebraic properties consistent with those imposed on the supporting complex.

A *pattern* on the complex K is a mapping π: $K \to J$. When J is a set of numbers with a zero it makes sense to use the notation $\pi^p(\sigma_p^i) = \pi(\sigma_p^i)$ and $\pi^p(\sigma_q^i) = 0$ for $p \neq q$, and the mapping π can be resolved into the 'submappings' π^p written

$$\pi = \pi^0 \oplus \pi^1 \oplus \dots \oplus \pi^N , \qquad N = \dim K .$$

The meaning of the symbol $\oplus$ will be left open for the time being. In the usual development of algebraic topology $\oplus$ corresponds to the direct sum of the graded chain modules. Here it can be interpreted as the usual + on the number system J.

The inner product notation
Elementary discussions of sets and mappings often disguise the fact that elements of sets can be used to define mappings of mappings. In other words the roles of elements and mappings can often be reversed, and often algebraic isomorphisms can be deduced. Such so-called duality is very common in mathematics and forms a central part of algebraic topology. The element-mapping distinction is often simply a matter of viewpoint, but notation such as $f(x)$ tends to disguise this.

The inner product notation as used in Q-analysis is defined as $(\sigma, \pi) = \pi(\sigma)$, so that (σ, π) is an element of the coefficient set J. The notation is not used to make things look more fancy or out of a perverse desire to confound by using unfamiliar notation, there are genuine advantages. The outstanding advantage is the clear presentation of duality and the ease of definitions which involve both the complex and its patterns.

Chain complexes and pattern polynomials
Let K be a simplicial complex with n simplices and dimension N, that is, $K = \{\sigma^1, \dots, \sigma^n\}$. Let J be the integers (in general J can be any ring) as the *coefficient ring*. Form the set $J \times K$ of terms $\alpha_i \sigma^i$ where α_i is a coefficient and σ^i is a simplex.

Let $c \in C(K)$, where $C(K)$ is the set of all formal linear combinations

$$c = \alpha_1 \sigma^1 + ... + \alpha_n \sigma^n = \sum_{i=1}^{n} \alpha_i \sigma^i .$$

In algebraic topology such a sum is called a *chain* of simplices, this use of the word chain being different from its use in the term chain of connection. Let $C(K)$ obey the addition rule

$$\sum_{i=1}^{n} \alpha_i \sigma^i + \sum_{i=1}^{n} \beta_i \sigma^i = \sum_{i=1}^{n} (\alpha_i + \beta_i) \sigma^i ,$$

then $C(K)$ is an additive abelian group (that is, a module) called the *chain module* of K over J with identity $\sum_{i=1}^{n} 0\sigma^i$ where 0 is the additive identity of J.

Let K_p be the set of simplices in K which have dimension exactly p. Form the submodules of $C(K)$ from $J \times K_p$ as follows. Let $C_p(K)$ be the set of all formal linear combinations

$$c_p = \alpha_1 \sigma_p^1 + ... + \alpha_x \sigma_p^x = \sum_{i=1}^{x} \alpha_i \sigma_p^i ,$$

where x is the number of p-simplices in K_p. If $C_p(K)$ satisfies the above addition rule it is called the *p-chain module* of K over J. In general the term *chain group* is also used for the term chain modules.

The chain module $C(K)$ can be written as the *direct sum* of the p-chain modules

$$C(K) = C_0(K) \oplus ... \oplus C_N(K)$$

where $N = \dim K$.

Atkin has shown how this dimensionally graded module can be made into an exterior algebra by defining the wedge operator Λ on the vertex set (Atkin, 1977). In the exterior algebra the chains of $C(K)$ are *polynomials*, and this term is sometimes used for the chains when the exterior algebra has not been explicitly constructed.

Pattern polynomials

Let π be a pattern on the complex K. Let $k_i = (\sigma^i, \pi)$. The polynomial given by

$$\sum_{i=1}^{n} k_i \sigma^i = k_1 \sigma^1 + ... + k_n \sigma^n , \qquad \text{for all } \sigma^i \in K ,$$

may be used to represent both the complex and the pattern π. By an abuse of language the symbol π is used in the literature both to represent the pattern as a mapping and to represent the above expression which is called a *pattern polynomial.*

A special pattern polynomial with $k_i = 1$ iff $\sigma^i \in K$, $k_i = 0$ otherwise, gives a representation of the complex K itself as the formal sum of every simplex in the complex.

Patterns as mappings and patterns as pattern polynomials have different properties which may confuse the unwary: a pattern gives a coefficient when applied as a mapping to a set of simplices, but a pattern polynomial is a set of simplices with numbers attached. Both interpretations of the term are useful in different circumstances and the interpretation of the symbol π is usually clear from context.

In an obvious way an operator + can be defined on the set of pattern polynomials as follows. Let π and ρ be pattern polynomials. The sum of π and ρ is defined as

$$\sum_{i \in I} (\sigma^i, \pi) \sigma^i + \sum_{i \in I'} (\sigma^i, \rho) \sigma^i = \sum_{i \in I \cup I'} [(\sigma^i, \pi) + (\sigma^i, \rho)] \sigma^i$$

$$= \sum_{i \in I \cup I'} (\sigma^i, \pi + \rho) \sigma^i \qquad \text{[by definition]} .$$

I and I' are the index sets defining the pattern polynomials π and ρ. By this definition the set of pattern polynomials is an additive abelian group.

The face operator

The face operator maps every p-simplex in a complex K to the sum of its $(p-1)$-dimensional faces. Formally the *face operator*, denoted f, is defined as

$$f\sigma_p = \sum_{\sigma^i_{p-1} \lesssim \sigma_p} \sigma^i_{p-1} .$$

The linear extension of f is defined as

$$f \sum \alpha_i \sigma^i = \sum \alpha_i f\sigma^i ,$$

so that

$$f^2\sigma_p = f(f\sigma_p) = f\Big(\sum_{\sigma^i_{p-1} \lesssim \sigma_p} \sigma^i_{p-1}\Big) = \Big[\sum_{\sigma^i_{p-1} \lesssim \sigma_p} \Big(\sum_{\sigma^j_{p-2} \lesssim \sigma^i_{p-1}} \sigma^j_{p-2}\Big)\Big]$$

$$= 2 \sum_{\sigma^j_{p-2} \lesssim \sigma_p} \sigma^j_{p-2} .$$

In general

$$f^r\sigma_p = r! \sum_{\sigma^i_{p-r} \lesssim \sigma_p} \sigma^i_{p-r} ,$$

which motivates the definition of the *exponential face operator* as

$$\hat{f}^r = \frac{1}{r!} f^r ,$$

so that applied to σ_p the result is

$$\hat{f}^r\sigma_p = \sum_{\sigma^i_{p-r} \lesssim \sigma_p} \sigma^i_{p-r} ,$$

which is the polynomial having each $p-r$ face of σ_p just once.

The coface operator

The *coface operator* maps p-dimensional pattern polynomials to $(p+1)$-dimensional pattern polynomials, $\Delta\colon \Pi^p \to \Pi^{p+1}$, and is defined as

$$(f\sigma_{p+1}, \pi^p) = (\sigma_{p+1}, \Delta\pi^p) ,$$

where $\pi^p \in \Pi^p$ and is a pattern on K, and $\sigma_{p+1} \in K$.

Δ^r is interpreted as an operator $\Delta^r\colon \Pi^p \to \Pi^{p+r}$ by the definition of f^r where

$$(f^r\sigma_{p+r}, \pi^p) = (f^{r-1}\sigma_{p+r}, \Delta\pi^p) = \ldots = (f^{r-s}\sigma_{p+r}, \Delta^s\pi^p) = \ldots = (\sigma_{p+r}, \Delta^r\sigma^p) ,$$

so that $\Delta^r\pi^p$ is a pattern on the $p+r$ dimensional simplices of K.

Like the face operator, the coface operator introduces unwanted factorial terms when applied many times. After this has been adjusted by the factor $1/r!$ one of the most important entities of Q-analysis is obtained: the *exponential coface operator*, $\hat{\Delta}$, is defined as

$$(\hat{f}\sigma_p, \pi^{p-1}) = (\sigma_p, \hat{\Delta}\pi^{p-1})$$

so that

$$(\hat{f}^r\sigma_{p+r}, \pi^p) = \frac{1}{r!}(f^r\sigma_{p+r}, \pi^p) = \frac{1}{r!}(\sigma_{p+r}, \Delta^r\pi^p) = (\sigma_{p+r}, \hat{\Delta}^r\pi^p)$$

whereby $\hat{\Delta}^r = (1/r!)\Delta^r$ for $r \geqslant 0$.

Changes in pattern values, t-forces

Atkin's definition of t-force relates changes in pattern value at dimension t to a pattern at dimension $t+1$ (Atkin, 1977). The need to consider changes in pattern values at both dimension t and $t+1$ has also been necessary in the study of q-transmission of changes (Johnson, 1981). The definition of t-force given here is part of the story, but it is simple and suffices for many applications. The term t-force is established in literature, but the equivalent definition of p-force is preferred here.

Let $\{\tau_0, \tau_1, \tau_2, ...\}$ be a sequence of 'points' in time (now times) where $\tau_i < \tau_j$ means τ_i is before τ_j, and this is assumed to be the case when $i < j$. Let the notation π_τ mean the pattern π on the complex K at time τ, and $\delta\pi^p_{\tau_i}$ will denote an *incremental change in* $\pi^p_{\tau_{i-1}}$ *with respect to* τ_i, where this is defined as

$$(\sigma_p, \delta\pi^p_{\tau_i}) = (\sigma_p, \pi^p_{\tau_i}) - (\sigma_p, \pi^p_{\tau_{i-1}}) .$$

$\delta\pi^p_{\tau_i}$ is called the p-dimension force, or *p-force* on the complex K between time τ_{i-1} and time τ_i. The number $(\sigma_p, \delta\pi^p_{\tau_i})$ is called the *value* of the force. A force is *attractive* iff its value is positive, it is *repulsive* iff its value is negative. The absolute value, that is, the unsigned value of a force is called its *magnitude*.

Backcloth and traffic

The fundamental tenet of Q-analysis is that the connectivity structure of a simplicial complex will play a determining role in the values patterns can take on the complex, and the way these patterns can change. Atkin has shown this to be the case in physics (Atkin, 1965; 1971) and the many examples in the literature support this view for social systems (Atkin, 1972; 1978).

In general changes in the value of a pattern depend both on the complex and on patterns defined on the conjugate. More generally many patterns on different complexes will be related to each other and the set of these complexes can be considered to be a relatively static *backcloth* which supports a *traffic* of activity represented by the patterns and their changes as t-forces. Thus, in general, the term backcloth refers to a set of simplicial complexes and the word traffic is the collective term for patterns and t-forces.

Whereas the time-space backcloth of physics is assumed to be unchanging, the backcloth for human activity appears to change over time. Much of social planning and administration involves changing the backcloth to prescribe the possible traffic, a sinister example being the fictional introduction of Newspeak; "It was intended that when Newspeak had been adopted once and for all and Oldspeak forgotten, a heretical thought—that is, a thought diverging from the principles of Ingsoc—should be literally unthinkable, at least so far as thought is dependent on words" (Orwell, 1949).

Strain pairs

Implicit in the definition of t-force is that the backcloth does not change with the changing pattern values. Atkin has likened this to a "framework under stress" (Atkin, 1974b) where the geometry of the framework is unchanged. The concept of *strain pair* is introduced to allow for a description of those cases in which the backcloth also changes.

Let P: $A \times B \times C \rightarrow \{\text{true, false}\}$ be an *open sentence* (Lipschutz, 1964), that is, given $a \in A$, $b \in B$, $c \in C$, P(a, b, c) is a proposition which can be judged either true or false by observation or deduction. Associated with P is a *cubic incidence matrix* M $= [M_{ijk}]$ where

$$M_{ijk} = \begin{cases} 1, & \text{iff } P(a_i, b_j, c_k) = \text{true} \\ 0, & \text{iff } P(a_i, b_j, c_k) = \text{false.} \end{cases}$$

Each $c_k \in C$ can be used to define an incidence matrix $\mathbf{M}_{c_k}$ by the rule

$$(M_{c_k})_{ij} = M_{ijk}$$

which can be considered the incidence matrix of a relation denoted by μ_{c_k}.

Let c and c' be elements of C. The incidence matrices $\mathbf{M}_c$ and $\mathbf{M}_{c'}$ can be constructed by the above rule, and these give the relations μ_c and $\mu_{c'}$. Every $a \in A$ determines a simplex in $K_A(B, \mu_c)$ which will be denoted $\sigma_r(a)_c$. The element a also determines the simplex $\sigma_s(a)_{c'}$ in $K_A(B, \mu_{c'})$. In general these two simplices are different. The polynomial expression $\sigma_s(a)_{c'} - \sigma_r(a)_c$ will be called a *strain pair* on a with respect to c' from c.

In the case C is a set of times, the strain pair represents a change in the structure associated with a between time c and time c'.

Consider a pattern π whose values also depend on the set C and use the notation π_c and $\pi_{c'}$ to represent π restricted to c and c', respectively. The notation of strain pair can be extended so that

$$(\sigma_s(a)_{c'}, \pi_{c'})\sigma_s(a)_{c'} - (\sigma_r(a)_c, \pi_c)\sigma_r(a)_c$$

is the *strain pair with respect to c' and c on a under π*. The absolute value $|(\sigma_s(a)_{c'}, \pi_{c'}) - (\sigma_r(a)_c, \pi_c)|$ will be called the *magnitude* of the strain.

Slicing, t-forces, and strain pairs

Let $\check{\mathbf{M}}$ be a matrix with arbitrary entries, and let $\check{\mathbf{M}}$ represent a weighted relation between the sets A and B. If A and B are finite there are a finite number of distinct incidence matrices which can be obtained from $\check{\mathbf{M}}$ by slicing. Let the set C index these incidence matrices and by implication the slicing procedures which produce them. This gives the incidence matrices $\mathbf{M}_c$ and relations μ_c, $c \in C$. The strain pair $\sigma_s(a)_{c'} - \sigma_r(a)_c$ is produced by changing from a view of the structure of $\mathbf{M}$ determined by slicing type c to a view of the structure of $\mathbf{M}$ determined by slicing type c'.

The meaning and/or legitimacy of slicing sometimes causes worry. From the mathematical point of view all slicing is legitimate, the procedure giving it meaning. From the point of view of applications the particular slicing is given meaning in terms of *thresholds*, that is, ranges of pattern values outside of which specified things cannot occur. For example one can define a relation between villages in India according to their separation being less than twelve kilometers. The structure sliced at twelve km (as opposed to, say fifteen km) is peculiarly relevant to those villagers who can walk up to twelve km to a market in a day. Although this structure may be different from that sliced at, say eleven km, it permits a clear analysis and interpretation of how a class of people determined by the 12 km slicing may behave as traffic on the 12 km structure (Johnson and Wanmali, 1981).

Slicing can be given a general interpretation in terms of traffic on the sliced structure. Given any matrix $\check{\mathbf{M}}$, the *finest slicing* is that which makes every nonzero entry a 1 in the incidence matrix and leaves all the zeros as they are. If A has m elements and B has n elements let K_A and K_B be the complexes obtained from the finest slicing. Then every column of $\check{\mathbf{M}}$ defines a pattern $\pi(i)$ on the named simplices of K_A:

$$(\sigma(a_i), \pi(j)) = \check{M}_{ij}, \qquad \text{for } i = 1, \ldots, m,$$

and every row of $\check{\mathbf{M}}$ defines a pattern $\Pi(i)$ on K_B:

$$(\sigma(b_j), \Pi(i)) = \check{M}_{ij}, \qquad \text{for } j = 1, \ldots, n.$$

Also every row of $\check{M}$ defines another graded pattern on K_A as $(\langle b_j \rangle, \pi^0(i)) = \check{M}_{ij}$ with $\pi^p(i) = \hat{\Delta}^p \pi^0(i)$, $i = 1, ..., n$; and every column of M defines another graded pattern on K_B as $(\langle a_i \rangle, \Pi^0(j)) = \check{M}_{ij}$ with $\Pi^q(j) = \hat{\Delta}^q \Pi^0(j)$, $j = 1, ..., m$.
As the slicing gets coarser and named simplices are sliced out of the complexes these patterns are automatically adjusted since it is implicitly assumed the patterns take value zero on any simplex not belonging to the complex on which they are defined. Thus slicing introduces a filtered set of strain pairs which 'clip out' the simplices and their pattern values, these originally being determined by the finest slicing.

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